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NORMAL MODE VIBRATIONS OF SYSTEMS OF
ELASTICALLY CONNECTED PARALLEL BEAMS

by

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ABSTRACT

The report presents the development and solution of the differential equations of motion of a system of n -elastically connected parallel beams. Using the results developed for the general n -beam system, the particular case of a two beam system was analyzed in detail. The frequencies and associated mode shapes for various support conditions are presented explicitly.

Vibration experiments were performed in order to ascertain the degree of applicability of the theory. It was found that at least up to the eighth mode, reasonably good agreement obtains.

INTRODUCTION

Flexural vibration of the single bar or beam has been extensively studied by many investigators. The validity of the Bernoulli-Euler theory has been studied and interesting generalizations of that useful theory have been developed. Little work has been done on structures built up from beams. A particular case of interest consists of a parallel system of thin beams elastically connected. The dynamics of such structural units is of interest particularly because of their possible use in various areas of technology, including that concerned with space platforms for use in orbital problems.

Preliminary to studies of dynamical response of such systems it is necessary to determine the normal mode shapes and their associated frequencies. It is the purpose of this report to provide explicitly these data and the experimentally determined basis for their validity.

SYSTEM OF DIFFERENTIAL EQUATIONS OF VIBRATION OF n-BEAMS

Using Bernoulli-Euler beam theory, the system of differential equations for n-elastically connected beams shown in Figure 1 can be written:

$$EI_1 \frac{\partial^4 w_1}{\partial x^4} + \rho_1 \frac{\partial^2 w_1}{\partial t^2} = -k_1(w_1 - w_2) \quad (1)$$

.....

$$EI_1 \frac{\partial^4 w_1}{\partial x^4} + \rho_1 \frac{\partial^2 w_1}{\partial t^2} = -k_{1-1}(w_1 - w_{1-1}) - k_1(w_1 - w_{1+1}) \quad (2)$$

.....

$$EI_n \frac{\partial^4 w_n}{\partial x^4} + \rho_n \frac{\partial^2 w_n}{\partial t^2} = -k_{n-1}(w_n - w_{n-1}) \quad (3)$$

According to the general theory of elasticity, the free vibrations of such a system are harmonic, so we may write for each w_n :

$$w_n = X_n(x)e^{i\omega t} \quad (4)$$

where ω is the natural frequency and $X_n(x)$ is a function of the space variable x only.

Using Eq. (4) the system of partial differential equations (1)-(3) reduce to the following system of ordinary differential equations

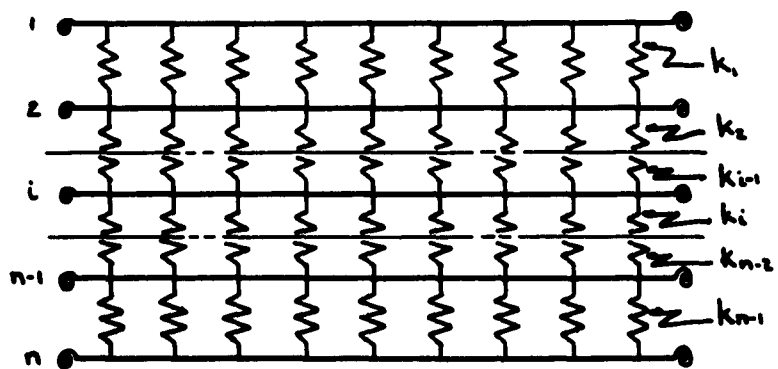


Figure 1. . System of n -Elastically
Connected Slender Beams

$$EI_1 \frac{d^4 X_1}{dx^4} - \rho_1 \omega^2 X_1 = -k_1(X_1 - X_2) \quad (5)$$

.....

$$EI_1 \frac{d^4 X_1}{dx^4} - \rho_1 \omega^2 X_1 = -k_{1-1}(X_1 - X_{1-1}) - k_1(X_1 - X_{1+1}) \quad (6)$$

.....

$$EI_n \frac{d^4 X_n}{dx^4} - \rho_n \omega^2 X_n = -k_{n-1}(X_n - X_{n-1}) \quad (7)$$

Solving these equations simultaneously, we obtain a single $4n^{\text{th}}$ order ordinary differential equation on say X_1

$$\sum_{n=1}^n \mu_n \frac{d^{4n} X_1}{dx^{4n}} + \mu_0 X_1 = 0 \quad (8)$$

where the μ_n are in general simple functions of the physical parameters of the problem and the frequency ω . Eq. (8) is obviously an ordinary differential with constant coefficients whose solution may be written as usual

$$X_1 = ce^{\lambda x} \quad (9)$$

Substituting Eq. (9) into Eq. (8) gives the characteristic equation

$$\sum_{n=0}^n \mu_n \lambda^{4n} = 0 \quad (10)$$

If we let

$$\beta = \lambda^4 \quad (11)$$

Eq. (10) becomes

$$\sum_{n=0}^n \mu_n \beta^n = 0 \quad (12)$$

Solving Eq. (12) for the β 's and using Eq. (11), all the λ 's are determined. The solution X_1 may now be written

$$X_1 = \sum_{m=1}^{4n} c_m e^{\lambda_m x} \quad (13)$$

In the solution of Eq. (12), if any of the β_n 's are zero or are repeated, the solution (13) to the differential equation will be modified according to the well known methods for ordinary differential equations with constant coefficients for those particular values of β_n .

If solution (13) is then substituted into the system of ordinary differential equations, we can get X_2, \dots, X_n as follows

$$X_j = c_{1j} e^{\lambda_1 x} \quad \begin{matrix} j = 1, 2, \dots, n \\ i = 1, 2, \dots, 4n \end{matrix} \quad (14)$$

The summation convention is used for the repeated index where the c_{1j} are linear functions of λ 's, ω 's and physical constants.

Any elastic rotational or elastic support boundary conditions may be investigated. For each of the beams in the n -beam system, there will be four associated boundary conditions.

Applying the $4n$ homogeneous boundary conditions we get a system of $4n$ homogeneous simultaneous equations in the constants c_1

$$a_{1s} c_1 = 0 \quad \begin{matrix} i = 1, \dots, 4n \\ s = 1, \dots, 4n \end{matrix} \quad (15)$$

where the a_{1s} are linear functions of λ 's, ω 's and physical constants.

In order to have non-trivial solutions, the determinant of the coefficients of the c_1 must vanish.

$$|a_{1s}| = 0 \quad (16)$$

The expansion of determinant (16) gives the frequency equation. For each natural frequency the c_1 can be determined in terms of one of the other c_1 from Eq. (15). The complete solution is therefore

$$w_j = c_{1j} e^{\lambda_1 x} e^{i\omega t} \quad \begin{matrix} i = 1, \dots, 4n \\ j = 1, \dots, n \end{matrix} \quad (17)$$

FREE VIBRATIONS OF ELASTICALLY CONNECTED DOUBLE BEAM SYSTEMS

As a preliminary to an experimental verification of the theory and to illustrate the general theory of n-beams, the case of a double beam system consisting of two identical beams elastically connected will be studied in detail. The equations of motion then are:

$$EI \frac{\partial^4 w_1}{\partial x^4} + \rho \frac{\partial^2 w_1}{\partial t^2} = -k(w_1 - w_2) \quad (18)$$

$$EI \frac{\partial^4 w_2}{\partial x^4} + \rho \frac{\partial^2 w_2}{\partial t^2} = -k(w_2 - w_1) \quad (19)$$

The solutions may be taken in the following form:

$$w_1 = X_1(x)e^{i\omega t} \quad (20)$$

$$w_2 = X_2(x)e^{i\omega t} \quad (21)$$

Substituting Eqs. (20) and (21) into Eqs. (18) and (19) give

$$EI \frac{d^4 X_1}{dx^4} - \rho\omega^2 X_1 = -k(X_1 - X_2) \quad (22)$$

$$EI \frac{d^4 X_2}{dx^4} - \rho\omega^2 X_2 = -k(X_2 - X_1) \quad (23)$$

Solving Eq. (22) for X_2 we get

$$x_2 = \frac{EI}{k} \frac{d^4 x_1}{dx^4} + \left(1 - \frac{\rho \omega^2}{k}\right) x_1 \quad (24)$$

and now substituting x_2 into Eq. (23) we get

$$\left(\frac{EI}{k}\right)^2 \frac{d^8 x_1}{dx^8} + \left(\frac{2EI}{k} - \frac{2EI}{k} \frac{\omega^2 \rho}{k}\right) \frac{d^4 x_1}{dx^4} + \left(\frac{\omega^4 \rho^2}{k^2} - \frac{2\omega^2 \rho}{k}\right) x_1 = 0 \quad (25)$$

Letting $\frac{EI}{k} = a$ and $\frac{\omega^2 \rho}{k} = b$, Eq. (25) becomes

$$\frac{d^8 x_1}{dx^8} + \left(\frac{2a - 2ab}{a^2}\right) \frac{d^4 x_1}{dx^4} + \frac{b^2 - 2b}{a^2} = 0 \quad (26)$$

All Ends Simply Supported

Using the procedure outlined for the n-beam case, we will now treat the case of two beams being simply supported. The boundary conditions for this case are

$$\begin{aligned} x_1(0) &= 0 & x_1(l) &= 0 \\ x_1''(0) &= 0 & x_1''(l) &= 0 \\ x_2(0) &= 0 & x_2(l) &= 0 \\ x_2''(0) &= 0 & x_2''(l) &= 0 \end{aligned} \quad (27)$$

From the boundary conditions (27) and using expression (24) we finally get the determinantal equation

1	1	1	1	0	0	0
λ_1^2	λ_2^2	λ_3^2	λ_4^2	0	0	0
λ_{1e}	λ_{2e}	λ_{3e}	λ_{4e}	0	0	0
$\lambda_{1e} \lambda_{2e}$	$\lambda_{2e} \lambda_{3e}$	$\lambda_{3e} \lambda_{4e}$	$\lambda_{4e} \lambda_{1e}$	0	0	0
0	0	0	0	1	1	1
0	0	0	0	λ_5^2	λ_6^2	λ_8^2
0	0	0	0	λ_{5e}	λ_{6e}	λ_{8e}
0	0	0	0	$\lambda_{5e} \lambda_{6e}$	$\lambda_{6e} \lambda_{7e}$	$\lambda_{7e} \lambda_{8e}$

Solving the determinantal equation (28) the frequencies and associated mode shapes are shown to be:

(a)

$$\omega^2 = \frac{EI}{\rho} \left(\frac{n\pi}{l} \right)^4 \quad (29)$$

$$X_1 = A' \sin \frac{n\pi x}{l} \quad (30)$$

$$X_2 = A' \sin \frac{n\pi x}{l}$$

(b)

$$\omega^2 = \frac{EI}{\rho} \left(\frac{n\pi}{l} \right)^4 + \frac{2k}{\rho} \quad (31)$$

$$X_1 = A' \sin \frac{n\pi x}{l} \quad (32)$$

$$X_2 = -A' \sin \frac{n\pi x}{l}$$

Following a procedure exactly analogous to the one outlined, the mode shapes and their associated frequencies for the following cases have been determined.

All Ends Free

(a)

$$\omega^2 = k_n^4 \frac{EI}{\rho}$$

where

$$\cos k_n l \cosh k_n l = 1$$

$$X_1 = A' \left\{ \frac{\cosh k_n x + \cos k_n x}{\cosh k_n l - \cos k_n l} - \frac{\sinh k_n x + \sin k_n x}{\sinh k_n l - \sin k_n l} \right\}$$

$$X_2 = X_1 \quad .$$

(b)

$$\omega^2 = k_n^4 \frac{EI}{\rho} + \frac{2k}{\rho}$$

where

$$\cos k_n l \cosh k_n l = 1$$

$$X_1 = A' \left\{ \frac{\cosh k_n x + \cos k_n x}{\cosh k_n l - \cos k_n l} - \frac{\sinh k_n x + \sin k_n x}{\sinh k_n l - \sin k_n l} \right\}$$

$$X_2 = -X_1 \quad .$$

As another possible solution each bar of the system may vibrate as a rigid body both angularly and transversely with frequency

$$\omega_0^2 = \frac{2k}{\rho} \quad .$$

There also exists the trivial solution where the system translates and rotates as a rigid body without vibration.

All Ends Fixed

(a)

$$\omega^2 = k_n^4 \frac{EI}{\rho}$$

where

$$\cos k_n l \cosh k_n l = 1$$

$$X_1 = A' \left\{ \frac{\sinh k_n (x - \frac{l}{2})}{\sinh \frac{k_n l}{2}} - \frac{\sin k_n (x - \frac{l}{2})}{\sin \frac{k_n l}{2}} \right\}$$

$$X_2 = X_1 \quad .$$

(b)

$$\omega^2 = k_n^4 \frac{EI}{\rho} + \frac{2k}{\rho}$$

where

$$\cos k_n l \cosh k_n l = -1$$

$$X_1 = A' \left\{ \frac{\sinh k_n (x - \frac{l}{2})}{\sinh \frac{k_n l}{2}} - \frac{\sin k_n (x - \frac{l}{2})}{\sin \frac{k_n l}{2}} \right\}$$

$$X_2 = -X_1 \quad .$$

Double Cantilever, Both Beams Fixed at 0

(a)

$$\omega^2 = k_n^4 \frac{EI}{\rho}$$

where

$$\cos k_n l \cosh k_n l = -1$$

$$X_1 = A' \left\{ \frac{\cosh k_n x - \cos k_n x}{\cosh k_n l + \cos k_n l} - \frac{\sinh k_n x - \sin k_n x}{\sinh k_n l + \sin k_n l} \right\}$$

$$X_2 = X_1$$

(b)

$$\omega^2 = k_n^4 \frac{EI}{\rho} + \frac{2k}{\rho}$$

where

$$\cos k_n l \cosh k_n l = -1$$

$$X_1 = A' \left\{ \frac{\cosh k_n x - \cos k_n x}{\cosh k_n l + \cos k_n l} - \frac{\sinh k_n x - \sin k_n x}{\sinh k_n l + \sin k_n l} \right\}$$

$$X_2 = -X_1$$

Double Propped Cantilever, Both Beams Fixed at End 1

(a)

$$\omega^2 = k_n^4 \frac{EI}{\rho}$$

where

$$\tanh k_n l = \tan k_n l$$

$$X_1 = A' \left\{ \frac{\sinh k_n x}{\sinh k_n l} - \frac{\sin k_n x}{\sin k_n l} \right\}$$

$$X_2 = X_1 \quad .$$

(b)

$$\omega^2 = k_n^4 \frac{EI}{\rho} + \frac{2k}{\rho}$$

where

$$\tanh k_n l = \tan k_n l$$

$$X_1 = A' \left\{ \frac{\sinh k_n x}{\sinh k_n l} - \frac{\sin k_n x}{\sin k_n l} \right\}$$

$$X_2 = -X_1 \quad .$$

All of these results are shown in tabular form in Figure 2.

One Bar Simply Supported, Other Bar Free

In all of the previously determined cases, there exists a symmetry between the boundary conditions for the two beams.

HINGED-HINGED				
HINGED-HINGED	$C_1 \pi^4$	$C_2 16 \pi^4$	$C_3 81 \pi^4$	$C_4 256 \pi^4$
HINGED-HINGED				
HINGED-HINGED	$C_1 \pi^4$	$C_2 16 \pi^4$	$C_3 81 \pi^4$	$C_4 256 \pi^4$
FREE-FREE				
FREE-FREE	$C_1 4.73^4$	$C_2 7.85^4$	$C_3 11.00^4$	$C_4 14.14^4$
FREE-FREE				
FREE-FREE	$C_1 4.73^4$	$C_2 7.85^4$	$C_3 11.00^4$	$C_4 14.14^4$
FIXED-FIXED				
FIXED-FIXED	$C_1 4.12^4$	$C_2 7.85^4$	$C_3 11.00^4$	$C_4 14.14^4$
FIXED-FIXED				
FIXED-FIXED	$C_1 4.13^4$	$C_2 7.85^4$	$C_3 11.00^4$	$C_4 14.14^4$
FIXED-FREE				
FIXED-FREE	$C_1 1.88^4$	$C_2 4.69^4$	$C_3 7.85^4$	$C_4 11.00^4$
FIXED-FREE				
FIXED-FREE	$C_1 1.88^4$	$C_2 4.69^4$	$C_3 7.85^4$	$C_4 11.00^4$
HINGED-FIXED				
HINGED-FIXED	$C_1 3.93^4$	$C_2 7.07^4$	$C_3 10.21^4$	$C_4 13.35^4$
HINGED-FIXED				
HINGED-FIXED	$C_1 3.93^4$	$C_2 7.07^4$	$C_3 10.21^4$	$C_4 13.35^4$

NODES ARE INDICATED AS A PROPORTION OF LENGTH L MEASURED FROM LEFT END
ELASTIC CONNECTORS BETWEEN BARS ARE NOT SHOWN

ANGULAR NATURAL FREQUENCY $\omega_1 = C_1 \sqrt{\frac{EI}{\rho L^3}}$; $\omega_2 = C_2 \sqrt{\frac{EI}{\rho L^3}} + \frac{2k}{L}$, RAD/SEC²

Figure 2. Modal Shapes and Natural Frequencies for Some Systems of Elastically Connected Double Beams

E = YOUNG'S MODULUS, LB/IN²

I = AREA MOMENT OF INERTIA OF BEAM CROSS SECTION, IN⁴

L = LENGTH OF BEAM, IN

ρ = MASS PER UNIT LENGTH OF BEAM, LB SEC²/IN³

k = MODULUS OF ELASTIC CONNECTORS, LB/IN³

C_n, C_n^0 = COEFFICIENTS FROM TABLE ABOVE, THE SUBSCRIPT n REFERS TO MODE NUMBER

Due to this, the 8 x 8 determinant was reduced to the evaluation of two 4 x 4 determinants. If however, this symmetry is not present, the zeros of the 8 x 8 determinant must be found. As an example the following case is shown.

The boundary conditions for one bar simply supported, the other free, are

$$\begin{aligned}
 x_1(0) &= 0 & x_1(l) &= 0 \\
 x_1''(0) &= 0 & x_1''(l) &= 0 \\
 x_2''(0) &= 0 & x_2''(l) &= 0 \\
 x_2'''(0) &= 0 & x_2'''(l) &= 0
 \end{aligned}
 \tag{33}$$

Using relationship (24) and the boundary conditions the following determinantal equation is obtained.

$$\begin{array}{cccccccc}
 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & \lambda_5^2 & \lambda_6^2 & \lambda_7^2 & \lambda_8^2 \\
 \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3 & -\lambda_5^3 & -\lambda_6^3 & -\lambda_7^3 & -\lambda_8^3 \\
 \lambda_{1e} & \lambda_{2e} & \lambda_{3e} & \lambda_{4e} & -\lambda_{5e} & -\lambda_{6e} & -\lambda_{7e} & -\lambda_{8e} \\
 \lambda_1^2 \lambda_{1e} & \lambda_2^2 \lambda_{2e} & \lambda_3^2 \lambda_{3e} & \lambda_4^2 \lambda_{4e} & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & \lambda_5^2 \lambda_{5e} & \lambda_6^2 \lambda_{6e} & \lambda_7^2 \lambda_{7e} & \lambda_8^2 \lambda_{8e} \\
 \lambda_1^3 \lambda_{1e} & \lambda_2^3 \lambda_{2e} & \lambda_3^3 \lambda_{3e} & \lambda_4^3 \lambda_{4e} & \lambda_5^3 \lambda_{5e} & \lambda_6^3 \lambda_{6e} & \lambda_7^3 \lambda_{7e} & \lambda_8^3 \lambda_{8e}
 \end{array}$$




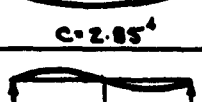
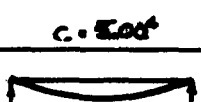
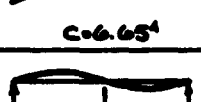


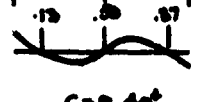
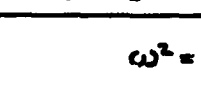
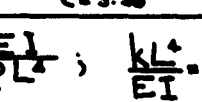
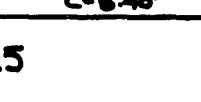
= 0 (34)

The zeros of determinant (34) are now found giving the natural frequencies. From these terms the mode shape associated with each natural frequency can be determined. For a specific case they are shown in Figure 3.

VIBRATION EXPERIMENTS WITH DOUBLE BEAM SYSTEM

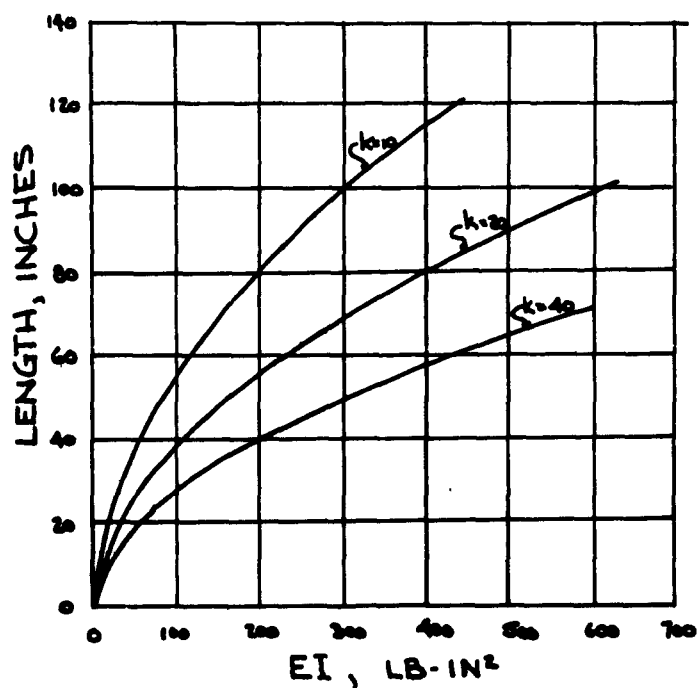
The experimental model used (See Figure 4) in the normal mode vibration experiments was supported on a massive reinforced concrete base 3' x 3' x 10'. The beam supports were made of 3/4" steel plates, in which provision was made for the placement of pin and roller supports. The pin support consisted of an actual pin placed through the bar at its mid height, while a roller support was provided by use of four roller bearings placed on the lateral boundaries of the bar, so as to provide a momentless end condition which would, however, allow axial motion of the bar. Tracks were placed above and below the model and were firmly attached to the steel support plates giving added rigidity to the supporting structure.

The model tested consisted of two 1/2" x 1/2" x 39.9" parallel bars, elastically connected with 20 coil springs placed on 2 inch centers. The bars were cut from standard cold rolled sections and had a Young's modulus close to 29×10^6 psi as determined by test. The springs were 2.0" in length and had a spring constant of 40 pounds per inch. They were affixed to the bars using Duco cement.

HINGED-HINGED			
FREE-FREE			
HINGED-HINGED			
FREE-FREE			

$$\omega^2 = C \frac{EI}{\rho L^4} ; \frac{KL^4}{EI} = 325$$

Modal Shapes and Natural Frequencies for
a Particular System



Curves for Determination of

k, L, EI for Case $\frac{KL^4}{EI} = 325$

Figure 3.

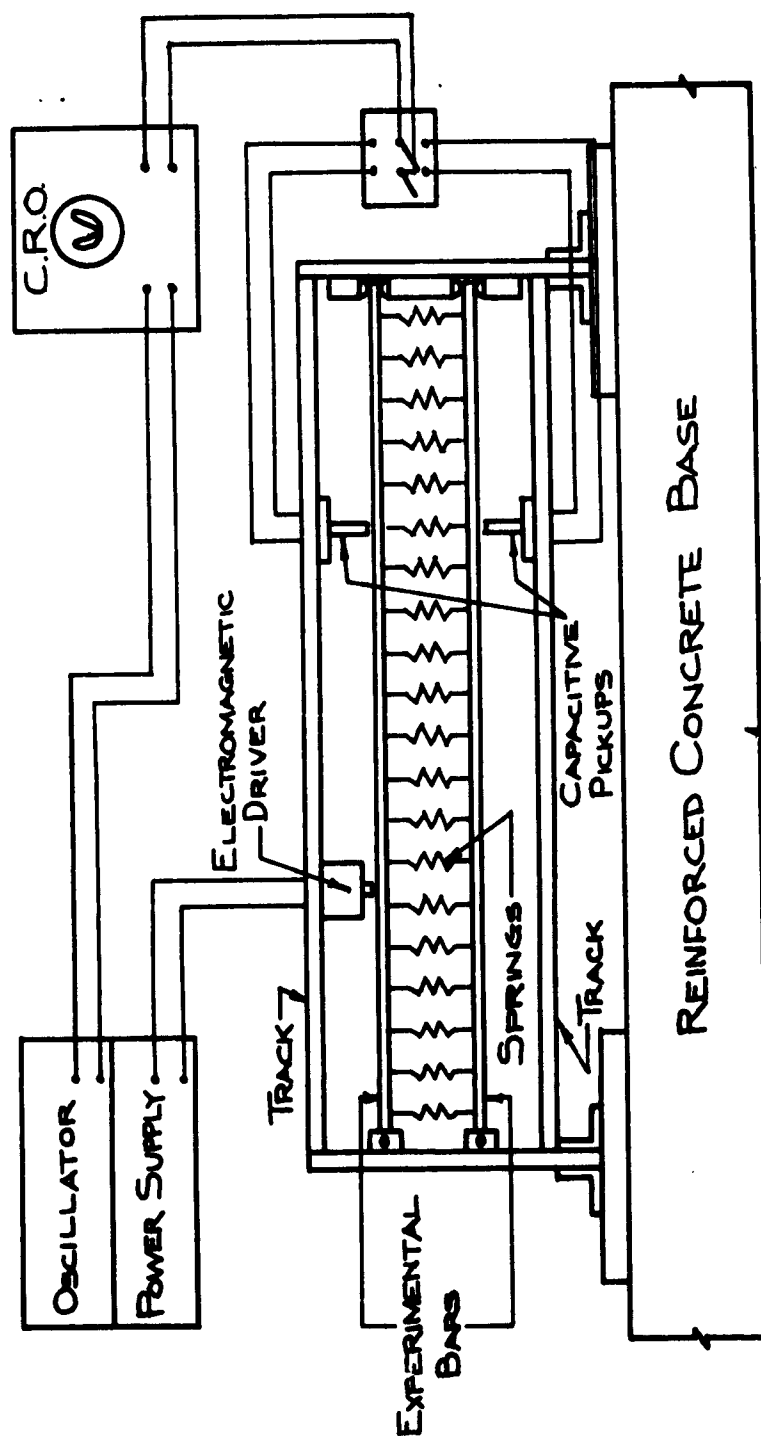


Figure 4. Experimental Apparatus for Determination of Frequencies of Vibration and Nodal Patterns

The system was driven by an electromagnet supplied¹ from an oscillator. The output of the oscillator was displayed on the upper beam of a Type 502 Tektronix Dual Beam Oscilloscope. The response of the model was obtained using capacitive type pickups² placed on the aforementioned tracks. This signal was displayed on the lower beam of the oscilloscope. Resonant frequencies were determined by examination of the capacitive pickup response as the frequency spectrum was traversed. A marked increase in response indicated a resonant frequency. Due to the pull-pull characteristic of the electromagnet, the frequency response of the beam system was twice the frequency set on the oscillator.

The upper beam vertical amplifier was then switched to the horizontal deflection plates and a figure 8 Lissajous figure resulted due to the 2:1 frequency ratios. The capacitive pickup was then moved on the track until the figure 8 horseshoe degenerated to a straight line. Continued movement of pickup inverted the figure 8 horseshoe. The nodal point was located when the straight line configuration was observed. Both bars were checked at the various resonant frequencies

¹W. H. Hoppmann II, N. J. Huffington, Jr., and L. S. Magness, "A Study of Orthogonally Stiffened Plates", J. Appl. Mech., 23, p. 343, (1956).

²S. N. Shafer and R. Plunkett, "A Miniature Oscilloscope and Vibration Pickup for Nodal Pattern Tracing", Proc. Soc. Exp. Stress Analysis, 13, No. 1, p. 123, (1955).

and all nodal points were recorded. In order to determine the phasing of the vibrating bars, the responses of two capacitive pickups, placed at corresponding locations of the bars, were displayed on the upper and lower oscilloscope beams. A difference switch on the Type 502 algebraically subtracted the lower beam input from the upper beam input. As one pickup was above the beam system and the other below it, the response when two points of the system were in phase resulted in a zero amplitude (straight line) sine wave, while when they were out of phase, a sine wave whose amplitude was the sum of the amplitudes of the original sine wave responses was observed.

The relative amplitudes of displacement were determined using micrometer barrels fitted with conical caps. The micrometers were electrically connected to a unit³ which, when contact was made by a micrometer with a bar, a clicking switch as well as a neon light would be activated. Initial readings on the micrometers which were set at various locations on the bars were made with the system at rest. The system was then driven at a resonant frequency. The micrometer heads were then made to just contact the vibrating bars and additional readings were made. By differencing these

³W. H. Hoppmann II, "Impulsive Loads on Beams", Proc. Soc. Exp. Stress Analysis, 10, No. 1, p. 157, (1952).

readings, the relative amplitudes of displacement were determined.

The mode shapes were drawn using the results of the nodal point, phasing, and relative amplitude measurements.

Three cases, each with different end supports, were tested. These were: Case I - all four boundaries simply supported; Case II - all boundaries free; Case III - upper beam simply supported, lower beam free. For Case II, light string was suspended from the upper track and was looped on the upper beam, at locations corresponding to the locations of nodal points for the particular resonant frequency being investigated, as support for the system. The location of the string supports at the nodal points did not change the free-free boundary conditions and, therefore, did not introduce any error.

As a check on the pin and roller supports, a single beam, simply supported, was driven and its resonant frequencies were determined. These values were checked against calculated theoretical values. The values were within a few percent of one another, indicating a reasonable approximation to simple support conditions.

NUMERICAL EXAMPLES AND EXPERIMENTAL RESULTS

In order to provide a basis for comparison between calculated results and experimental results. the double beam system

chosen corresponded to those used as models in the experimental work.

For purposes of illustration, three cases were investigated. These were: Case I - all four boundaries simply supported; Case II - all boundaries free; Case III - upper beam simply supported, lower beam free.

Using the formulae shown in Figure 2 and Figure 3, the frequencies in Tables 1, 2, and 3 were computed. The experimental frequencies shown in Tables 1, 2, and 3 were determined by the procedure outlined in previous section.

Case I - All Four Boundaries
Simply Supported

Calc	Exp
v.p.s.	
28	29
79	79
113	112
135	137
253	244
264	261
451	456
460	490

Table 1

Case II - All Boundaries Free

Calc v.p.s.	Exp
65	64
98	97
178	173
192	191
350	334
358	354
578	574
583	590

Table 2

Case III - Upper Beam Simply
Supported, Lower Beam Free

Calc v.p.s.	Exp
23	23
49	48
72	70
84	84
128	126
205	182

Table 3

For Case III, if Figure 3 is used for combinations of I , EI and k , the same values of c will be found as for the particular case investigated.

The location of nodal points for all cases investigated checked to within 2% of their calculated values.

CONCLUSIONS AND DISCUSSION

Theory has been developed for the free vibrations of n-elastically connected parallel beams. As a consequence the normal mode shapes and associated natural frequencies can be calculated for any combination of homogeneous boundary conditions. For a very small number of beams the calculations can be conducted with simple computational devices. However, for a larger number of beams it is clear that high speed computers will be required.

For the double beam system the frequencies and normal mode shapes have been determined for several interesting boundary conditions. Results have been tabulated and compared with those obtained from experiments on steel beams coupled with steel springs.

It is concluded that for the lower modes, at least up to the eighth, the agreement between theory and experiment is very good.